The Parabolic Probability Distribution
Building Our Distribution

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September, 2023

In this white paper we will create our own probability distribution via a simple polynomial equation. To that end we will work through the following hypothetical problem...

Our Hypothetical Problem

Our client owns shares in a startup where the current share price is $25.00. We are tasked with calculating our current share price multiple at some future time T given the following go-forward assumptions...

Table 1: Modeling Assumptions

Assumption	Value	Notes
Probability that multiple = 0.00	0.30	If multiple is zero then startup fails at time T.
Probability that 0.00 < multiple ≤ 2.00	0.20	Arbitrary data point in $[0, 6]$ range.
Probability that 2.00 < multiple ≤ 6.00	0.50	Maximum price at time T is six times current price.

Question 1: Graph the cumulative probability distribution of our share price muliple at time T.
Question 2: Graph the probability density function when $x > 0$.
Question 3: Prove that our parabola is monotonically increasing.
Question 4: What is the mean and variance of our multiple at time T?

Parabola Definitions

We will define our parabola $f(x)$ to be the following third-order polynomial...

$$f(x) = a\,x^3 + b\,x^2 + c\,x + d \;\text{ ...where... a, b, c, and d are real numbers and } a \neq 0 \tag{1}$$

Note that our parabola will be subject to the following contraints...

$$\text{Constraints: } 0 \leq x \leq w \;\text{ ...and... } f(x) \text{ is monotonically increasing} \tag{2}$$

Using Equation (1) above, the equations for the first and second derivatives of our parabola $f(x)$ with respect to the independent variable x are...

$$f'(x) = 3\,a\,x^2 + 2\,b\,x + c \;\text{ ...and... } f''(x) = 6\,a\,x + 2\,b \tag{3}$$

We will define our parabola such that it passes through the following three data points on the x:f(x) plane...

$$\text{start point: } x = 0, f(x) = h \;\text{ ...and... mid point: } x = m, f(x) = n \;\text{ ...and... end point: } x = w, f(x) = 1 \tag{4}$$

We will define our parabola such that the first derivative of $f(x)$ at data point $x = w, f(x) = 1$ is a small, positive number. Using Equation (3) above, this statement in equation form is...

$$\text{slope of parabola at end point} = f'(w) = g \;\text{ ...where... } g \text{ is a small positive number} \tag{5}$$

To define parameter values a, b, c and d we will solve the following system of simultaneous equations...

Table 2: Simultaneous Equations

Description	Data point	Equation
Start point	$0, h$	$a \times 0^3 + b \times 0^2 + c \times 0^1 + d = h$
Mid point	m, n	$a \times m^3 + b \times m^2 + c \times m^1 + d = n$
End point	$w, 1$	$a \times w^3 + b \times w^2 + c \times w^1 + d = 1$
Slope at end point	$w, 1$	$3 \times a \times w^2 + 2 \times b \times w^1 + c = g$

Using the data in the table above, we will define the following matrix and vectors...

$$\mathbf{A} = \begin{bmatrix} 0^3 & 0^2 & 0^1 & 1 \\ m^3 & m^2 & m^1 & 1 \\ w^3 & w^2 & w^1 & 1 \\ 3\,w^2 & 2\,w & 1 & 0 \end{bmatrix} \text{ ...and... } \vec{\mathbf{b}} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \text{ ...and... } \vec{\mathbf{v}} = \begin{bmatrix} h \\ n \\ 1 \\ g \end{bmatrix} \tag{6}$$

Using the matrix and vector definitions in Equation (6) above, we can rewrite our system of simultaneous equations as the following matrix:vector product...

$$\mathbf{A}\,\vec{\mathbf{b}} = \vec{\mathbf{v}} \tag{7}$$

The solution to Equation (7) above is...

$$\mathbf{A}^{-1}\,\vec{\mathbf{v}} = \vec{\mathbf{b}} \text{ ...where... vector } \vec{\mathbf{b}} \text{ contains the parameter values a, b, c and d} \tag{8}$$

To ensure that our parabola $f(x)$ is monotonically increasing, we will calculate the value of the inpendent variable x such that first derivative of $f(x)$ is equal to zero (i.e. parabola changes direction). If the value of x is in the range $[0, w]$ then $f(x)$ is not monotonically increasing. To calculate the value of x we iterate the following Newton-Raphson equation until x and $\hat{x}$ are approximately equal...

$$\text{new } \hat{x} = \hat{x} + \frac{f'(x) - f'(\hat{x})}{f''(\hat{x})} \text{ ...where... } \hat{x} = \text{estimate of actual } x \text{ ...and... } f'(\text{actual } x) = 0 \tag{9}$$

Parabolic Probability Distribution

Given that parabola value at any point on the x-axis range $[0, w]$ is non-negative, parabola value is monotonically increasing, and parabola value at $x = w$ is equal to one, our parabola as defined above is a valid cumulative probability distribution. Using Equation (3) above, the equation for the probability density function is...

$$PDF(x) = f(x) = d \text{ when } x = 0 \;\Big|\; PDF(x) = f'(x) = 3\,a\,x^2 + 2\,b\,x + c \text{ when } x \neq 0 \tag{10}$$

Using Equations (4) and (10) above, the sum of incremental probabilities over the entire x-axis range $[0, w]$ is...

$$\text{Cumulative probability over } [0, w] = f(w) = d + \int\limits_0^w PDF(x)\,\delta x = 1.00 \tag{11}$$

Using Appendix Equation (21) below, the equation for the first moment of our probability distribution is...

$$FM = \mathbb{E}\Bigg[x\Bigg] = \frac{3}{4}\,a\,w^4 + \frac{2}{3}\,b\,w^3 + \frac{1}{2}\,c\,w^2 \tag{12}$$

Using Appendix Equation (23) below, the equation for the second moment of our probability distribution is...

$$SM = \mathbb{E}\Bigg[x^2\Bigg] = \frac{3}{5}\,a\,w^5 + \frac{2}{4}\,b\,w^4 + \frac{1}{3}\,c\,w^3 \tag{13}$$

Using Equations (12) and (13) above, the equation for the probability distribution mean and variance are...

$$\text{mean} = FM \text{ ...and... variance} = SM - FM^2 \tag{14}$$

The Answers To Our Hypothetical Problem

Question 1: Graph the cumulative probability distribution of the share price muliple at time T.

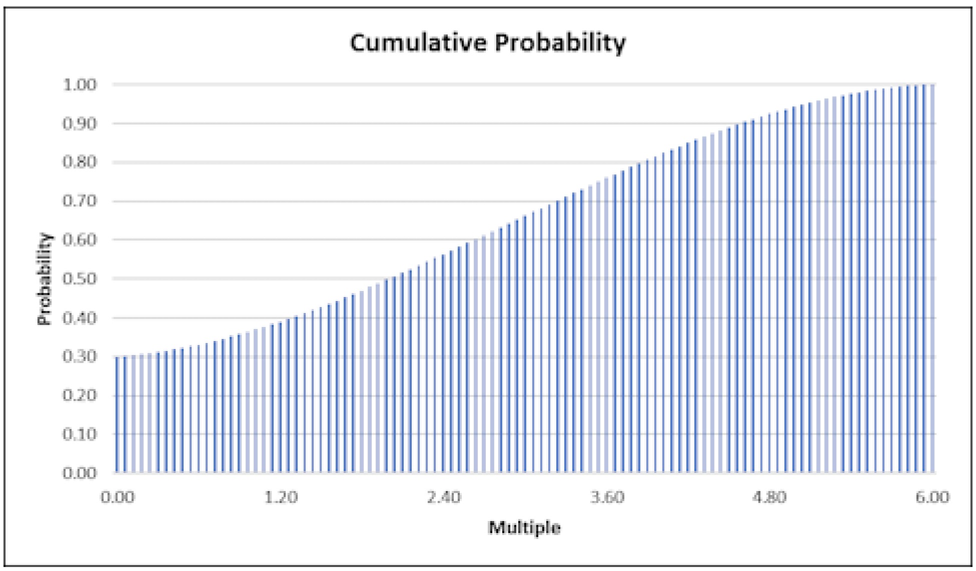
Using our model assumptions in Table 1 above and the simultaneous equations in Table 2 above, our matrix and vectors via Equation (6) above are...

$$\mathbf{A} = \begin{bmatrix} 0^3 & 0^2 & 0^1 & 1 \\ 2^3 & 2^2 & 2^1 & 1 \\ 6^3 & 6^2 & 6^1 & 1 \\ 3\times 6^2 & 2\times 6^1 & 1 & 0 \end{bmatrix} \text{ ...and... } \vec{\mathbf{b}} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \text{ ...and... } \vec{\mathbf{v}} = \begin{bmatrix} 0.30 \\ 0.30+0.20 \\ 0.30+0.20+0.50 \\ 0.01 \end{bmatrix} \tag{15}$$

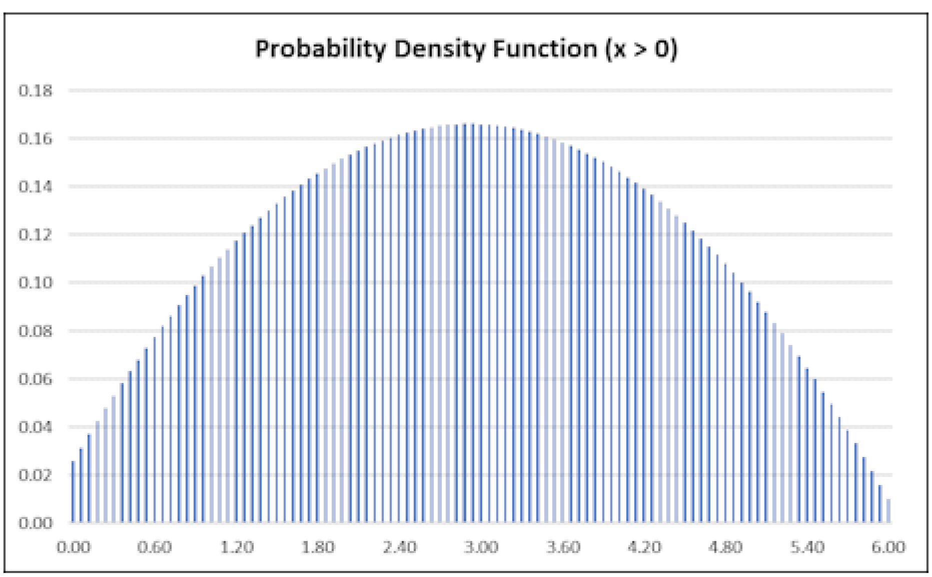
Using the matrix and vectors in Equation (15) above, the parameters for our parabola are...

$$\mathbf{A}^{-1}\,\vec{\mathbf{v}} = \vec{\mathbf{b}} = \begin{bmatrix} -0.0055 \\ 0.0481 \\ 0.0258 \\ 0.3000 \end{bmatrix} \tag{16}$$

The graph of our parabola using the parameter estimates in Equation (16) above is...



Question 2: Graph the probability density function when $x > 0$.



Question 3: Prove that our parabola is monotonically increasing.

xhat	f'(x)	f'(xhat)	f"(xhat)
3.00	0.00	0.17	-0.0026
65.92	0.00	-65.16	-2.0738
34.50	0.00	-16.25	-1.0395
18.87	0.00	-4.02	-0.5250
11.21	0.00	-0.97	-0.2729
7.67	0.00	-0.21	-0.1565
6.36	0.00	-0.03	-0.1132
6.11	0.00	0.00	-0.1049
6.10	0.00	0.00	-0.1046
6.10	0.00	0.00	-0.1046

Note that our guess value (row one) is $w \div 2 = 3.00$.

Since the value of x via the Newton-Raphson iteration above ($x = 6.10$) is not in the range $[0, 6]$ then our parabola is monotonically increasing.

Question 4: What is the mean and variance of our multiple at time T?

Using Equations (12) and (16) above, the equation for the first moment of our distribution is...

$$FM = \frac{3}{4} \times -0.0055 \times 6^4 + \frac{2}{3} \times 0.0481 \times 6^3 + \frac{1}{2} \times 0.0258 \times 6^2 = 2.0525 \tag{17}$$

Using Equations (13) and (16) above, the equation for the second moment of our distribution is...

$$SM = \frac{3}{5} \times -0.0055 \times 6^5 + \frac{2}{4} \times 0.0481 \times 6^4 + \frac{1}{3} \times 0.0258 \times 6^3 = 7.4040 \tag{18}$$

Using Equations (14), (17) and (18) above, the answer to the question is...

$$\text{mean} = 2.0525 \text{ ...and... } \text{variance} = 7.4040 - 2.0525^2 = 3.1912 \tag{19}$$

Appendix

A. We want to solve the following integral...

$$\begin{aligned} I &= \int_0^w PDF(x)\, x\, \delta x \\ &= \int_0^w \Big(3\, a\, x^2 + 2\, b\, x + c\Big) x\, \delta x \\ &= \int_0^w \Big(3\, a\, x^3 + 2\, b\, x^2 + c\, x\Big) \delta x \\ &= 3\, a \int_0^w x^3 \delta x + 2\, b \int_0^w x^2 \delta x + c \int_0^w x\, \delta x \end{aligned} \tag{20}$$

The solution to the integral in Equation (20) above is...

$$I = \frac{3}{4}\, a\, x^4 \Big[_0^w + \frac{2}{3}\, b\, x^3 \Big[_0^w + \frac{1}{2}\, c\, x^2 \Big[_0^w = \frac{3}{4}\, a\, w^4 + \frac{2}{3}\, b\, w^3 + \frac{1}{2}\, c\, w^2 \tag{21}$$

B. We want to solve the following integral...

$$
\begin{aligned}
I &= \int_0^w PDF(x)\, x^2\, \delta x \\
&= \int_0^w \left(3\, a\, x^2 + 2\, b\, x + c\right) x^2\, \delta x \\
&= \int_0^w \left(3\, a\, x^4 + 2\, b\, x^3 + c\, x^2\right) \delta x \\
&= 3\, a \int_0^w x^4 \delta x + 2\, b \int_0^w x^3 \delta x + c \int_0^w x^2\, \delta x
\end{aligned}
\tag{22}
$$

The solution to the integral in Equation (20) above is...

$$
I = \frac{3}{5}\, a\, x^5 \Big[_0^w + \frac{2}{4}\, b\, x^4 \Big[_0^w + \frac{1}{3}\, c\, x^3 \Big[_0^w = \frac{3}{5}\, a\, w^5 + \frac{2}{4}\, b\, w^4 + \frac{1}{3}\, c\, w^3 \tag{23}
$$